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Nonintegrability of the two-body problem in constant curvature spaces

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Abstract
We consider the reduced two-body problem with the Newton and the oscillator potentials on the sphere $\mathbf{S}^2$ and the hyperbolic plane $\mathbf{H}^2$. For both types of interaction we prove the nonexistence of an additional meromorphic integral for the complexified dynamic systems.

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1. Introduction

The study of mechanics on constant curvature spaces began in the nineteenth century after the rise of non-Euclidean geometry [1–3]. Similarly to the Euclidean case in constant curvature simply connected spaces (the sphere $\mathbf{S}^n$ and the hyperbolic space $\mathbf{H}^n$) there are two exceptional central potentials V_N and V_o (below Newton and oscillator potentials). They have some nice properties, which can be grounds for their definitions.

Both these potentials make all bounded trajectories of a one-body problem closed [4]. Moreover, these trajectories (bounded and unbounded) are conics [5], which can be naturally defined in constant curvature spaces [6–8].

The one-body motion in the Newton potential satisfies the analogues of the three Kepler laws [2, 3, 5]. This potential is also the fundamental solution of the Laplace equation. The corresponding force in the hyperbolic space was already proposed by Lobachevski (in 1835–1838) [9] and Bolyai (between 1848 and 1851) [10] as the value $F(\rho)$ which is inverse to the area of the sphere in $\mathbf{H}^3$ of radius ρ with an attractive body in the centre. These results can be considered as predecessors of general relativity. After the rise of this theory the above-mentioned papers were almost completely forgotten.

Similar models later attracted attention from the point of view of quantum mechanics and the theory of integrable dynamical systems. This led to the rediscovery of results described above in many papers, sometimes with partial improvements, see for example [11, 12]. Note however that the almost forgotten results of Killing and Liebmann were described in the survey [13]. The corresponding quantum mechanical problems in constant curvature spaces were studied in [14–18] and other papers.

The two-body problem with a central interaction in constant curvature spaces $\mathbf{S}^n$ and $\mathbf{H}^n$ considerably differs from its Euclidean analogue. The variable separation for the latter problem is trivial, while for the former one no central potentials are known that admit a variable separation.

The two-body problem with a central interaction in constant curvature spaces was considered for the first time in [19]. In Euclidean space this problem is reduced to a one-body problem in a central potential after separating the centre of mass motion. Due to the absence of Galilei transformations, the situation for the constant curvature spaces is different. The two-body problem is invariant with respect to the isometry group, but for non-Euclidean space this group is not wide enough to imply the integrability of this problem in any sense.

The natural problem of finding central potentials corresponding to integrable two-body problems is currently far from its solution. This can be explained by the fact that the existing methods of the theory of integrable and nonintegrable dynamical systems do not work in the presence of a functional parameter.

As a limiting case of a two-body problem in constant curvature spaces, one can consider the restricted two-body problem: the 'heavy' body moves with a constant velocity along a geodesic, while the 'light' one moves in a potential of a 'heavy' body.

The nonintegrability of this problem with the potential V_N and V_o on the sphere $\mathbf{S}^2$ was proved in [20, 21] in the class of meromorphic functions. Similar results with smaller restrictions, valid also for the restricted two-body problem on the hyperbolic plane $\mathbf{H}^2$, were obtained in [22].

Here, we prove the nonexistence of an additional meromorphic first integral for the restricted two-body problem on the spaces $\mathbf{S}^2$ and $\mathbf{H}^2$ using the Morales–Ramis theory [23].

2. Reduced two-body problems

Note that the classical two-body problem on $\mathbf{S}^n$ and $\mathbf{H}^n$ reaches its full generality at $n = 3$ [19]. Its Hamiltonian reduction to the system with two degrees of freedom was carried out in [19] ($n = 3$) and in [24] ($n = 2$) by explicit coordinate calculations. A more conceptual approach to this reduction was derived in [25].

Here, we shall use the following description of the reduced dynamical systems for $n = 2$, combining approaches from [19] and [24–26].

2.1. The reduced two-body problem on the sphere $\mathbf{S}^2$

Let $\mathbf{S}^2$ be the sphere of the radius R with the standard metric. The configuration space for the two-body problem on $\mathbf{S}^2$ is $Q = (\mathbf{S}^2 \times \mathbf{S}^2)\backslash\mathrm{diag}$. Let $Q_{\mathrm{op}} \simeq \mathbf{S}^2$ be a subset of Q, consisting of pairs of opposite points. The phase space T^*Q can be represented as

$$T^*Q = (T^*I \times T^*\,\mathrm{SO}(3)) \cup \widetilde{T}^*Q_{\mathrm{op}},$$

where $I = (0, \pi R)$ and $\widetilde{T}^*Q_{\mathrm{op}}$ is the restriction of the cotangent bundle $T^*(Q \times Q)$ onto Q_{op}.

The space $\widetilde{T}^*Q_{\mathrm{op}}$ is the submanifold in T^*Q of the codimension 2; therefore, a typical trajectory does not intersect it. Below we consider only such trajectories.

The group SO(3) acts by symplectomorphisms on the second factor of the product

$$M = T^*I \times T^*\,\mathrm{SO}(3),$$

endowed with the standard symplectic structure of a cotangent bundle. Therefore, the reduced phase space for M has the form [27]

$$\widetilde{M} = T^*I \times \mathcal{O},$$

where $\mathcal{O}$ is a SO(3) orbit w.r.t. the coadjoint action in the space $\mathfrak{so}^*(3)$ dual to the Lie algebra $\mathfrak{so}(3)$. The orbit $\mathcal{O}$ is endowed with the Kirillov symplectic form.

The Killing form on the Lie algebra $\mathfrak{so}(3)$ generates its natural identification with the dual space $\mathfrak{so}^*(3)$ and makes both these spaces Euclidean. The coadjoint orbits in $\mathfrak{so}^*(3)$ are standard spheres in the Euclidean space $\mathbf{E}^3$ with the common centre $0 \in \mathfrak{so}^*(3)$ and the Kirillov symplectic form on them coincides with area forms, generated by the Euclidean structure.

The reduced Hamiltonian function on $\widetilde{M}$ for the two-body problem is

$$h_s = \frac{(1+r^2)^2}{8mR^2}\left(p_r^2 + \frac{p_2^2}{r^2}\right) + \frac{(1+r^2)p_r p_0 + \gamma^2}{2m_1R^2} - \frac{p_2^2}{m_1R^2} + \frac{(1-r^2)p_1p_2}{2m_1R^2r} + V(r). \tag{2.1}$$

Here, p_i are orthogonal coordinates in $\mathfrak{so}^*(3)$; $m := m_1m_2/(m_1+m_2)$ is the reduced mass for body masses $m_1, m_2, r = \tan(\rho/(2R))$, ρ is the distance between the bodies, p_r is the momentum corresponding to the coordinate r and the orbit $\mathcal{O} \equiv \mathcal{O}_\gamma$ is defined by the equation

$$p_0^2 + p_1^2 + p_2^2 = \gamma^2, \qquad \gamma \geqslant 0.$$

The Poisson brackets for variables r, p_r, p_0, p_1, p_2 are as follows,

$$\begin{array}{llll} \{r, p_r\} = 0, & \{p_0, p_1\} = -p_2, & \{p_1, p_2\} = -p_0, & \\ \{p_2, p_0\} = -p_1, & \{r, p_i\} = 0, & \{p_r, p_i\} = 0, & i = 0, 1, 2 \end{array} \tag{2.2}$$

and the evolution of any smooth function $f = f(r, p_r, p_0, p_1, p_2)$ is defined by the equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, h\}. \tag{2.3}$$

At every fixed moment of time, the momentum p_2 corresponds to the rotation of the second body around the first one, the momentum p_0 corresponds to the motion of bodies along the geodesic connecting them and the momentum p_1 corresponds to the motion of the system in the direction, normal to this geodesic.

In the exceptional case $\gamma = 0$, it holds $\mathcal{O}_0 = \mathrm{pt}$; the reduced system has only one degree of freedom and it corresponds to the motion of the bodies along a common geodesic with the null value of the total momentum.

Let $\gamma > 0$ and p_2, φ be cylinder coordinates on $\mathcal{O}_\gamma$ such that

$$p_0 = \sqrt{\gamma^2 - p_2^2}\sin\varphi, \qquad p_1 = \sqrt{\gamma^2 - p_2^2}\cos\varphi.$$

Then it holds $\varphi = \arctan(p_0/p_1)$ and $\{\varphi, p_2\} = 1$. Thus $p_\varphi = p_2, \varphi$ are canonical coordinates on $\mathcal{O}_\gamma$ (with singularities at the points $p_2 = \pm\gamma$) and the Hamiltonian function (2.1) can be written in the form

$$h_s = \frac{(1+r^2)^2}{8mR^2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - \frac{p_\varphi^2}{m_1R^2} + \frac{\sqrt{\gamma^2 - p_\varphi^2}}{2m_1R^2} \times \left(p_\varphi\frac{1-r^2}{r}\cos\varphi + (1+r^2)p_r\sin\varphi\right) + V(r) + \frac{\gamma^2}{2m_1R^2}, \tag{2.4}$$

coinciding with formula (7) in [24] (cf also formula (18) in [19] for $\alpha = \gamma, \beta = 0$).

The Hamiltonian function (2.1) can be represented in another form after the substitution of the variables r, p_r by a new pair of canonical ones θ, p_θ such that

$$\theta = \frac{\rho}{R}, \qquad p_\theta = \frac{1}{2}(1+r^2)p_r, \qquad \{\theta, p_\theta\} = 1.$$

This substitution leads to the following expression:

$$h_s = \frac{1}{2mR^2}\left(p_\theta^2 + \frac{p_2^2}{\sin^2\theta}\right) + \frac{p_\theta p_0}{m_1R^2} - \frac{p_2^2}{m_1R^2} + \frac{p_1p_2}{m_1R^2}\cot\theta + \frac{\gamma^2}{2m_1R^2} + V(\theta)$$
$$= h_{s,1} + h_{s,2} + \frac{\gamma^2}{2m_1R^2} + V(\theta), \qquad (2.5)$$

where

$$h_{s,1} := \frac{1}{2mR^2}\left(p_\theta^2 + \frac{p_2^2}{\sin^2\theta}\right), \qquad h_{s,2} := \frac{p_\theta p_0}{m_1R^2} - \frac{p_2^2}{m_1R^2} + \frac{p_1p_2}{m_1R^2}\cot\theta.$$

Below we shall use expression (2.5) for the Hamiltonian function h_s though all claims can be reformulated for other coordinate systems.

The Hamiltonian function h_s corresponds to trivially integrable Hamiltonian systems in the following three cases.

(1) The free motion of bodies: $V(\theta) = 0$. One can easily verify that $\{h_{s,1}, h_{s,2}\} = 0$ and an additional integral in this case is $h_{s,1}$ or $h_{s,2}$.
(2) $m_1 = \infty$. This case corresponds to the motion of the second body in a central potential of the fixed first body. An additional integral in this case is p_2.
(3) $m_2 = \infty$ or equivalently $m = m_1$. This case corresponds to the motion of the first body in a central potential of the fixed second one. In this case the function h_s can be represented in the form

$$h_s = \frac{1}{2m_1R^2}(p_\theta + p_0)^2 + \frac{(p_1\sin\theta + p_2\cos\theta)^2}{2m_1R^2\sin^2\theta} + V(\theta).$$

Since $\{p_1\sin\theta + p_2\cos\theta, p_\theta + p_0\} = 0$, an additional integral in this case is $p_1\sin\theta + p_2\cos\theta$.

One can also derive from (2.5) the Hamiltonian function for the restricted two-body problem. Let the first body be a 'heavy' one ($m_1 \to \infty$) and it moves along a geodesic Γ with a constant velocity ω. Since the momentum p_2 corresponds to the rotation of the second body around the first one, it holds $p_2/m_1 \to 0$. Let ψ be an angle between the geodesic Γ and the geodesic Υ connecting the bodies. Since the momentum p_0 corresponds to the motion of the bodies along the geodesic Υ, then $p_0/m_1 \to \omega\cos\psi$. At last the momentum p_1 corresponds to the motion of the system in the direction perpendicular to Υ; therefore, $p_1/m_1 \to \omega\sin\psi$. Also one should omit the term $\gamma^2/(2m_1R^2)$, which is independent of time and tends to infinity, since $\gamma/m_1 \to \text{const} \neq 0$. This corresponds to the infinite kinetic energy of the 'heavy' body as $m_1 \to \infty$.

Thus from (2.5) one gets the Hamiltonian function for the restricted two-body problem

$$h_{s,r} = \frac{1}{2m_2R^2}\left(p_\theta^2 + \frac{p_2^2}{\sin^2\theta}\right) + \frac{\omega}{R^2}(p_\theta\cos\psi + p_2\sin\psi\cot\theta) + V(\theta).$$

Besides, since

$$\left\{\frac{p_0}{m_1}, p_2\right\} \to \omega\{\cos\psi, p_2\} = -\omega\sin\psi\{\psi, p_2\} \qquad \text{and} \qquad \left\{\frac{p_0}{m_1}, p_2\right\} = \frac{p_1}{m_1} \to \omega\sin\psi$$

one gets in the limiting case $\{\psi, p_2\} = -1$. Hence, the variables $p_\psi := -p_2, \psi$ are canonical and we obtain

$$h_{s,r} = \frac{1}{2m_2R^2}\left(p_\theta^2 + \frac{p_\psi^2}{\sin^2\theta}\right) + \frac{\omega}{R^2}(p_\theta\cos\psi - p_\psi\sin\psi\cot\theta) + V(\theta)$$

that coincides up to notations with corresponding expressions from [20, 22].

It holds

$$\left\{p_\theta^2 + \frac{p_\psi^2}{\sin^2\theta},\, p_\theta\cos\psi - p_\psi\sin\psi\cot\theta\right\} = 0$$

that corresponds to the integrability of the free motion.

Note that the Newton and the oscillator potentials mentioned in the introduction have the following forms:

$$V_N = -\alpha\cot\theta, \qquad V_o = \frac{\beta}{2}\tan^2\theta, \qquad \alpha, \beta = \text{const}, \qquad \alpha, \beta > 0. \tag{2.6}$$

Our main result for the spherical case is the following theorem.

Theorem 2.1. *The complexified Hamiltonian system with the Hamiltonian function (2.1) and potentials (2.6) does not admit an additional meromorphic first integral in the case* $m_1 \neq m$, $m_1 m\alpha\beta\gamma \neq 0$, *where* $p_0^2 + p_1^2 + p_2^2 = \gamma^2$.

2.2. The reduced two-body problem on the hyperbolic plane $\mathbf{H}^2$

Let $\mathbf{H}^2$ be the hyperbolic plane with a sectional curvature $-1/R^2$. The configuration space for the two-body problem on $\mathbf{H}^2$ is $Q = (\mathbf{H}^2 \times \mathbf{H}^2)\setminus\text{diag}$. Here, there are no opposite points and the phase space $M := T^*Q$ can be represented as

$$M = T^*\mathbb{R}_+ \times T^*\mathrm{O}_0(1,2),$$

where $\mathbb{R}_+ := (0, \infty)$ and $\mathrm{O}_0(1,2)$ is the identity component of the isometry group for $\mathbf{H}^2$. The Lie algebra for the Lie group $\mathrm{O}_0(1,2)$ is $\mathfrak{so}(1,2)$.

The reduced phase space for M is

$$\widetilde{M} = T^*\mathbb{R}_+ \times \mathcal{O},$$

where $\mathcal{O}$ is a $\mathrm{O}_0(1,2)$ orbit w.r.t. the coadjoint action in the space $\mathfrak{so}^*(1,2)$. It is endowed with the Kirillov symplectic form.

The Lie algebra $\mathfrak{so}(1,2)$, endowed with the Killing form, is the Minkowski space with the signature $(1,2)$ and can be naturally identified with its dual space $\mathfrak{so}^*(1,2)$. Let p_0, p_1, p_2 be orthogonal coordinates in $\mathfrak{so}^*(1,2)$ w.r.t. the Killing form Kil $:= p_0^2 + p_1^2 - p_2^2$. The orbits $\mathcal{O} \in \mathfrak{so}^*(1,2)$ are of the following types:

(1) $\mathcal{O}_0 = (0)$;
(2) $\mathcal{O}_c = \big((p_0, p_1, p_2) | p_0^2 + p_1^2 - p_2^2 = 0, (p_0, p_1, p_2) \neq (0,0,0)\big)$;
(3) $\mathcal{O}_\gamma = \big((p_0, p_1, p_2) | p_0^2 + p_1^2 - p_2^2 = \gamma,\big)\ \gamma > 0$;
(4) $\mathcal{O}_\gamma = \big((p_0, p_1, p_2) | p_0^2 + p_1^2 - p_2^2 = \gamma,\big)\ \gamma < 0$.

The orbit $\mathcal{O}_c$ is a cone without its vertex, the orbit $\mathcal{O}_\gamma$ for $\gamma > 0$ is a one-sheet hyperboloid and for $\gamma < 0$ it is a two-sheet hyperboloid.

The two-body Hamiltonian function on $\widetilde{M}$ can be obtained from (2.1) by the formal substitution

$$p_0 \to \mathrm{i}p_0, \quad p_1 \to \mathrm{i}p_1, \quad p_2 \to p_2, \quad p_r \to \mathrm{i}p_r, \quad r \to -\mathrm{i}r, \quad R \to \mathrm{i}R,$$

where i is the complex unit. This leads to the expression

$$h_h = \frac{(1-r^2)^2}{8mR^2}\left(p_r^2 + \frac{p_2^2}{r^2}\right) + \frac{(1-r^2)p_r p_0 + \gamma}{2m_1R^2} + \frac{p_2^2}{m_1R^2} + \frac{(1+r^2)p_1p_2}{2m_1R^2 r} + V(r), \tag{2.7}$$

where $r := \tanh(\rho/(2R))$; $\rho \in \mathbb{R}_+$ is the distance between the bodies, and $p_0^2 + p_1^2 - p_2^2 = \gamma$ on the orbit $\mathcal{O}_\gamma$.

The Poisson brackets for variables r, p_r, p_0, p_1, p_2 are as follows:

$$\{r, p_r\} = 0, \qquad \{p_0, p_1\} = p_2, \qquad \{p_1, p_2\} = -p_0,$$
$$\{p_0, p_2\} = p_1, \qquad \{r, p_i\} = 0, \qquad \{p_r, p_i\} = 0, \qquad i = 0, 1, 2.$$

One can define canonical variables p_φ, φ on $\mathcal{O}_\gamma \neq \mathrm{pt}$ by the formulae

$$p_0 = \sqrt{\gamma + p_2^2}\sin\varphi, \qquad p_1 = \sqrt{\gamma + p_2^2}\cos\varphi, \qquad p_2 = p_\varphi.$$

Then one gets the expression

$$h_h = \frac{(1-r^2)^2}{8mR^2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + \frac{p_\varphi^2}{m_1R^2} + \frac{\sqrt{\gamma + p_\varphi^2}}{2m_1R^2} \times \left(p_\varphi \frac{1+r^2}{r}\cos\varphi + (1-r^2)p_r\sin\varphi\right) + V(r) + \frac{\gamma}{2m_1R^2} \tag{2.8}$$

that up to notations coincides with formulae (11)–(13) from [24].

Let us also define a new pair of canonical variables θ, p_θ such that

$$\theta = \frac{\rho}{R}, \qquad p_\theta = \frac{1}{2}(1-r^2)p_r, \qquad \{\theta, p_\theta\} = 1.$$

Then, one gets from (2.7)

$$h_h = \frac{1}{2mR^2}\left(p_\theta^2 + \frac{p_2^2}{\sinh^2\theta}\right) + \frac{p_\theta p_0}{m_1R^2} + \frac{p_2^2}{m_1R^2} + \frac{p_1p_2}{m_1R^2}\coth\theta + \frac{\gamma}{2m_1R^2} + V(\theta). \tag{2.9}$$

Cases of a trivial integrability of the reduced two-body problem on the hyperbolic plane are similar to those described above for the spherical case and correspond to one of the equalities: $V(\theta) = 0$, $m_1 = \infty$ or $m = m_1$.

The Newton and the oscillator potentials now have the following forms:

$$V_N = -\alpha\coth\theta, \qquad V_o = \frac{\beta}{2}\tanh^2\theta, \qquad \alpha, \beta = \mathrm{const}, \qquad \alpha, \beta > 0. \tag{2.10}$$

The main result of the present paper for the hyperbolic case is as follows.

Theorem 2.2. *The complexified Hamiltonian system with the Hamiltonian function (2.7) and potentials (2.10) does not admit an additional meromorphic first integral in the case* $m_1 \neq m$, $m_1 m\alpha\beta\gamma \neq 0$, $\gamma > 0$, *where* $p_0^2 + p_1^2 - p_2^2 = \gamma$.

3. The result of the Morales–Ramis theory

Here we present the result from the Morales–Ramis theory [23], which will be used for proving the absence of an additional meromorphic integral for the reduced two-body problem on $\mathbf{S}^2$ and $\mathbf{H}^2$ with potentials (2.6) and (2.10).

Let M be a complex analytic manifold ($\dim_{\mathbb{C}} M = n$) and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v(x), \qquad t \in \mathbb{C}, \qquad x \in M \tag{3.1}$$

be a system of differential equations, where v is a holomorphic vector field on M. Let $x = \varphi(t)$ be a particular nonconstant solution of (3.1) and Γ be the Riemann surface in M, defined by the maximal analytic continuation of $\varphi(t)$. The *variational equations* along $\varphi(t)$ have the form

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = V(v)\xi, \xi \in T_\Gamma M, \tag{3.2}$$

where $T_\Gamma M$ is the restriction of the tangent bundle TM onto Γ and

$$V(v) := \frac{\partial v}{\partial x}(\varphi(t))$$

is a morphism $T_\Gamma M \to T_\Gamma M$ of the vector bundle $T_\Gamma M$.

Let system (3.1) be a Hamiltonian one with a Hamiltonian function H; in particular, n is even. Then the order of (3.2) can be reduced by 2. Indeed, let M_ε be a submanifold in M defined as $M_\varepsilon := (x \in M | H(x) = \varepsilon)$, $\varepsilon = \text{const}$; $N := T_\Gamma M_\varepsilon / T\Gamma$ be the normal bundle of the surface Γ in TM_ε and $\pi : T_\Gamma M_\varepsilon \to N$ be the canonical projection. Note that $\dim_{\mathbb{C}} N = n - 2$. Then,

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \pi[V(v)(\pi^{-1}\eta)], \qquad \eta \in N \tag{3.3}$$

is a well-defined system of differential equations on N since $V(v)\,(T\Gamma) \subset T\Gamma$ and $\pi(T\Gamma)$ is the null section of N. System (3.3) is called the *normal variational equations*.

Let $\mathcal{G}$ be the differential Galois group [28, 29] for system (3.3), i.e. a matrix group acting on fundamental solutions of (3.3) which does not change polynomial relations between them. Let also $\mathcal{G}_0$ be the identity component for $\mathcal{G}$.

Theorem 3.1 (see [23], theorem 4.1)**.** *Suppose that there are $n/2$ meromorphic first integrals of a Hamiltonian system (3.1) that are in involution and are independent in some neighbourhood of Γ. Then $\mathcal{G}_0$ is an abelian group.*

4. Particular solutions and variational equations

In order to simplify notations one can multiply the Hamiltonian functions (2.5) and (2.9) by m_1R^2 that is equivalent to changing the scale of the time axis and omit the constant summand. Thus one gets

$$h_s = \frac{1}{2\mu}\left(p_\theta^2 + \frac{p_2^2}{\sin^2\theta}\right) + p_\theta p_0 - p_2^2 + p_1 p_2 \cot\theta + V(\theta), \tag{4.1}$$

$$h_h = \frac{1}{2\mu}\left(p_\theta^2 + \frac{p_2^2}{\sinh^2\theta}\right) + p_\theta p_0 + p_2^2 + p_1 p_2 \coth\theta + V(\theta), \tag{4.2}$$

where $\mu := m/m_2 = m_1/(m_1 + m_2) \neq 0$.

Consider Hamiltonian systems with Hamiltonian functions (4.1) and (4.2) on reduced manifolds $\widetilde{M}_\gamma = T^*I \times \mathcal{O}_\gamma$ in the spherical case and $\widetilde{M}_\gamma = T^*\mathbb{R}_+ \times \mathcal{O}_\gamma$ in the hyperbolic case. For any potential $V(\theta)$ there are trajectories defined by the equalities $p_0 = p = \text{const} \neq 0$, $p_1 = p_2 = 0$. They correspond to the bodies motion along a common geodesic. For the spherical case all nondegenerate manifolds $\widetilde{M}_\gamma$, $\gamma > 0$ contain such a trajectory and for the hyperbolic case only those with $\gamma = p^2 > 0$. Denote the maximal analytic continuation of this trajectory by Γ in accordance with section 3.

One can choose p_1 and p_2 as local coordinates in a neighbourhood of Γ. Then using (2.2) and (2.3) one gets the normal variational equations in the spherical case:

$$\begin{aligned}\frac{\mathrm{d}p_1}{\mathrm{d}t} &= -p\cot\theta p_1 + \left(2p + p_\theta - \frac{p}{\mu\sin^2\theta}\right)p_2,\\ \frac{\mathrm{d}p_2}{\mathrm{d}t} &= -p_\theta p_1 + p\cot\theta p_2,\end{aligned} \tag{4.3}$$

where $p_\theta = p_\theta(t)$; $\theta = \theta(t)$ is a solution of the Hamiltonian system with the Hamiltonian function

$$h_0 = \frac{1}{2\mu} p_\theta^2 + p p_\theta + V(\theta) = \frac{1}{2\mu} (p_\theta + \mu p)^2 + V(\theta) - \frac{\mu}{2} p^2. \tag{4.4}$$

The normal variational equations in the hyperbolic case are

$$\begin{aligned} \frac{\mathrm{d}p_1}{\mathrm{d}t} &= -p \coth\theta p_1 - \left(2p + p_\theta + \frac{p}{\mu \sinh^2\theta}\right) p_2, \\ \frac{\mathrm{d}p_2}{\mathrm{d}t} &= -p_\theta p_1 + p \coth\theta p_2, \end{aligned} \tag{4.5}$$

where again $p_\theta = p_\theta(t)$, $\theta = \theta(t)$ is a solution of the Hamiltonian system with the Hamiltonian function (4.4).

One can compare these normal variational systems with their analogues for the restricted two-body problem from [20] and [22]. For example, in the spherical case the latter system can be written as

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = -\omega \cot\theta p_1 + \frac{p_2}{\sin^2\theta}, \qquad \frac{\mathrm{d}p_2}{\mathrm{d}t} = \omega p_\theta p_1 + \omega \cot\theta p_2. \tag{4.6}$$

The key factor for the determination of a differential Galois group for a system of linear differential equations is its reducibility to a system with rational coefficients. For the Newton and oscillator potentials, such a reduction for systems (4.3), (4.5) and (4.6) is possible.

4.1. The Newton potential

For the Newton potential $V = V_N = -\alpha \cot\theta$, system (4.6) becomes a Fuchsian one w.r.t. the independent variable p_θ that was found in [20]. This fact is also valid for system (4.3).

Indeed, denote $z := (p_\theta + \mu p)/\alpha$. One can easily check that system (4.3) can be written as

$$p_1'(z) = A(z) p_1 + B(z) p_2 \qquad p_2'(z) = C(z) p_1 - A(z) p_2 \tag{4.7}$$

with respect to the independent variable z, where

$$A(z) = \frac{p f(z)}{1 + f^2(z)}, \qquad B(z) = \frac{p}{\mu} - \frac{\alpha z + (2-\mu) p}{1 + f^2(z)},$$
$$C(z) = \frac{\alpha z - \mu p}{1 + f^2(z)}, \qquad f(z) = \frac{\alpha z^2}{2\mu} - \varepsilon$$

and the trajectory Γ corresponds to the equation

$$\frac{\alpha^2 z^2}{2\mu} - \alpha \cot\theta = \alpha\varepsilon = \text{const}.$$

System (4.7) is Fuchsian (see the appendix) with five regular singular points $z_{1,2} = \pm\varkappa$, $z_{3,4} = \pm\lambda$, $z_5 = \infty$, where

$$\varkappa := \sqrt{\frac{2\mu}{\alpha}(\varepsilon + \mathrm{i})} \notin \mathbb{R}, \qquad \lambda := \sqrt{\frac{2\mu}{\alpha}(\varepsilon - \mathrm{i})} \notin \mathbb{R}, \qquad \text{for} \quad \varepsilon \in \mathbb{R}.$$

We shall express all coefficients through four parameters $p, \mu, \varkappa, \lambda$. In particular it holds

$$f(z) = \mathrm{i}\frac{2z^2 - \lambda^2 - \varkappa^2}{\varkappa^2 - \lambda^2}, \qquad f^2(z) + 1 = 2\mathrm{i}\frac{(z^2 - \lambda^2)(z^2 - \varkappa^2)}{\varkappa^2 - \lambda^2}.$$

One can transform (4.7) into the linear differential equation for $p_2(z)$ of the second order

$$p_2''(z) = \frac{C'}{C} p_2' + \left(\frac{C'}{C} A + A^2 + CB - A'\right) p_2, \tag{4.8}$$

and then into equation (A.2) for the function $y(z) := p_2(z)/\sqrt{C}$, where

$$r(z) = \frac{C'}{C}A + A^2 + CB - A' - \frac{1}{2}\left(\frac{C'}{C}\right)' + \frac{1}{4}\left(\frac{C'}{C}\right)^2.$$

For the evaluation of the function $r(z)$ one can use computer analytical calculations, which lead to

$$r(z) = \sum_{j=1}^{4}\left(\frac{\alpha_j}{(z-z_j)^2} + \frac{\beta_j}{z-z_j}\right) + \frac{3}{4(z-z_0)^2} = \frac{3}{4z^2} + O\left(\frac{1}{z^3}\right) \quad \text{as} \quad z \to \infty, \tag{4.9}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{1-\mu}{64\varkappa^2}(p(\mu-1)(\lambda^2-\varkappa^2) + 4\mathrm{i}\varkappa(\mu+1))(p(\lambda^2-\varkappa^2)+4\mathrm{i}\varkappa),\\
\alpha_2 &= \frac{1-\mu}{64\varkappa^2}(p(\mu-1)(\lambda^2-\varkappa^2) - 4\mathrm{i}\varkappa(\mu+1))(p(\lambda^2-\varkappa^2)-4\mathrm{i}\varkappa),\\
\alpha_3 &= \frac{1-\mu}{64\lambda^2}(p(\mu-1)(\varkappa^2-\lambda^2) - 4\mathrm{i}\lambda(\mu+1))(p(\varkappa^2-\lambda^2)-4\mathrm{i}\lambda),\\
\alpha_4 &= \frac{1-\mu}{64\lambda^2}(p(\mu-1)(\varkappa^2-\lambda^2) + 4\mathrm{i}\lambda(\mu+1))(p(\varkappa^2-\lambda^2)+4\mathrm{i}\lambda),\\
\beta_1 &= \frac{\mu-1}{64(\varkappa^2-\lambda^2)\varkappa^3}((\mu-1)(5\varkappa^2-\lambda^2)(\varkappa^2-\lambda^2)^2p^2 - 32\mathrm{i}\mu(\varkappa^2-\lambda^2)\varkappa^3p\\
&\qquad - 16(\mu+1)\varkappa^2(3\varkappa^2+\lambda^2)),\\
\beta_2 &= \frac{\mu-1}{64(\varkappa^2-\lambda^2)\varkappa^3}((1-\mu)(5\varkappa^2-\lambda^2)(\varkappa^2-\lambda^2)^2p^2 - 32\mathrm{i}\mu(\varkappa^2-\lambda^2)\varkappa^3p\\
&\qquad + 16(\mu+1)\varkappa^2(3\varkappa^2+\lambda^2)),\\
\beta_3 &= \frac{\mu-1}{64(\varkappa^2-\lambda^2)\lambda^3}((1-\mu)(5\lambda^2-\varkappa^2)(\varkappa^2-\lambda^2)^2p^2 + 32\mathrm{i}\mu(\varkappa^2-\lambda^2)\lambda^3p\\
&\qquad + 16(\mu+1)\lambda^2(3\lambda^2+\varkappa^2)),\\
\beta_4 &= \frac{\mu-1}{64(\varkappa^2-\lambda^2)\lambda^3}((\mu-1)(5\lambda^2-\varkappa^2)(\varkappa^2-\lambda^2)^2p^2 + 32\mathrm{i}\mu(\varkappa^2-\lambda^2)\lambda^3p\\
&\qquad - 16(\mu+1)\lambda^2(3\lambda^2+\varkappa^2)),\ z_0 = \frac{\mu p}{\alpha} = \frac{p(\varkappa^2-\lambda^2)}{4\mathrm{i}}.
\end{aligned} \tag{4.10}$$

For $\mu = 1$ the expression for $r(z)$ is very simple

$$r(z) = \frac{3}{4(z-z_0)^2}. \tag{4.11}$$

Lemma 4.1. *Suppose that $\alpha, \varepsilon, \mu, p \in \mathbb{R}$, $\mu \neq 0, 1$, $p \neq 0$ and*

$$\left(\sqrt{\varepsilon^2+1} - \varepsilon\right)(\varepsilon^2+1) \neq \frac{(\mu-1)^2p^2}{4\alpha\mu}, \tag{4.12}$$

then $\alpha_i \notin \mathbb{R}$ for $i = 1, 2, 3, 4$.

Proof. Direct calculations imply $\alpha_1 = \frac{\mu^2-1}{4} - \frac{\mu-1}{4\alpha} p\mu^2\left(\frac{2}{\varkappa} + \frac{p(1-\mu)}{\varkappa^2\alpha}\right)$ and

$$\frac{2}{\varkappa} + \frac{p(1-\mu)}{\varkappa^2\alpha} = \frac{\sqrt{2\alpha}}{\sqrt{\mu}\sqrt{\varepsilon^2+1}}\left(\sqrt{\varepsilon - \mathrm{i}} + \frac{(\mu-1)p\mathrm{i}}{2\sqrt{2\alpha\mu}\sqrt{\varepsilon^2+1}}\right) + \frac{(1-\mu)p\varepsilon}{2\mu(\varepsilon^2+1)}.$$

Therefore $\alpha_1 \notin \mathbb{R}$ iff

$$\operatorname{Im}\sqrt{\varepsilon - \mathrm{i}} = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{\varepsilon^2+1} - \varepsilon} \neq \frac{(1-\mu)p}{2\sqrt{2\alpha\mu}\sqrt{\varepsilon^2+1}}$$

that is equivalent to (4.12). The consideration for α_2, α_3 and α_4 is similar. □

In the hyperbolic case, system (4.5) again is reduced to the Fuchsian system (4.7), where now

$$A(z) = \frac{pf(z)}{f^2(z)-1}, \qquad B(z) = \frac{p}{\mu} + \frac{\alpha z + (2-\mu)p}{f^2(z)-1},$$
$$C(z) = \frac{\alpha z - \mu p}{f^2(z)-1}, \qquad f(z) = \frac{\alpha z^2}{2\mu} - \varepsilon.$$

The trajectory Γ corresponds here to the equation

$$\frac{\alpha z^2}{2\mu} - \coth\theta = \varepsilon.$$

In this case the singular points are $z_{1,2} = \pm\varkappa$, $z_{3,4} = \pm\lambda$, $z_5 = \infty$ for

$$\varkappa := \sqrt{\frac{2\mu}{\alpha}(\varepsilon+1)}, \qquad \lambda := \sqrt{\frac{2\mu}{\alpha}(\varepsilon-1)}.$$

Again for the function $y(z) = p_2(z)/\sqrt{C(z)}$ one gets equation (A.2) with $r(z)$ given by (4.9), where

$$\begin{aligned}
\alpha_1 &= \frac{\mu-1}{64\varkappa^2}(p(\mu-1)(\varkappa^2-\lambda^2) - 4\varkappa(\mu+1))(p(\varkappa^2-\lambda^2) - 4\varkappa),\\
\alpha_2 &= \frac{\mu-1}{64\varkappa^2}(p(\mu-1)(\varkappa^2-\lambda^2) + 4\varkappa(\mu+1))(p(\varkappa^2-\lambda^2) + 4\varkappa),\\
\alpha_3 &= \frac{\mu-1}{64\lambda^2}(p(\mu-1)(\varkappa^2-\lambda^2) - 4\lambda(\mu+1))(p(\varkappa^2-\lambda^2) - 4\lambda),\\
\alpha_4 &= \frac{\mu-1}{64\lambda^2}(p(\mu-1)(\varkappa^2-\lambda^2) + 4\lambda(\mu+1))(p(\varkappa^2-\lambda^2) + 4\lambda),\\
\beta_1 &= \frac{\mu-1}{64(\varkappa^2-\lambda^2)\varkappa^3}((\mu-1)(\lambda^2-5\varkappa^2)(\varkappa^2-\lambda^2)^2p^2 + 32\mu(\varkappa^2-\lambda^2)\varkappa^3p\\
&\quad - 16(\mu+1)\varkappa^2(3\varkappa^2+\lambda^2)),\\
\beta_2 &= \frac{\mu-1}{64(\varkappa^2-\lambda^2)\varkappa^3}((\mu-1)(5\varkappa^2-\lambda^2)(\varkappa^2-\lambda^2)^2p^2 + 32\mu(\varkappa^2-\lambda^2)\varkappa^3p\\
&\quad + 16(\mu+1)\varkappa^2(3\varkappa^2+\lambda^2)),\\
\beta_3 &= \frac{\mu-1}{64(\varkappa^2-\lambda^2)\lambda^3}((\mu-1)(5\lambda^2-\varkappa^2)(\varkappa^2-\lambda^2)^2p^2 - 32\mu(\varkappa^2-\lambda^2)\lambda^3p\\
&\quad + 16(\mu+1)\lambda^2(3\lambda^2+\varkappa^2)),\\
\beta_4 &= \frac{\mu-1}{64(\varkappa^2-\lambda^2)\lambda^3}((\mu-1)(\varkappa^2-5\lambda^2)(\varkappa^2-\lambda^2)^2p^2 - 32\mu(\varkappa^2-\lambda^2)\lambda^3p\\
&\quad - 16(\mu+1)\lambda^2(3\lambda^2+\varkappa^2)), \; z_0 = \frac{\mu p}{\alpha} = \frac{p(\varkappa^2-\lambda^2)}{4}.
\end{aligned} \tag{4.13}$$

For $\mu = 1$ the expression for $r(z)$ coincides with (4.11).

Lemma 4.2. *Suppose that* $\alpha, \varepsilon, \mu, p \in \mathbb{R}$, $\mu \neq 0, 1$; $p \neq 0$, $\varepsilon < -1$, *then* $\alpha_i \notin \mathbb{R}$ *for* $i = 1, 2, 3, 4$.

Proof. Evidently, $\varkappa, \lambda \in \mathrm{i}\mathbb{R}\backslash(0)$ and one gets

$$\mathrm{i}\,\mathrm{Im}\,\alpha_1 = -\mathrm{i}\,\mathrm{Im}\,\alpha_2 = \frac{\mu^2(1-\mu)p}{2\alpha\varkappa} \neq 0, \qquad \mathrm{i}\,\mathrm{Im}\,\alpha_3 = -\mathrm{i}\,\mathrm{Im}\,\alpha_4 = \frac{\mu^2(1-\mu)p}{2\alpha\lambda} \neq 0. \qquad \square$$

4.2. The oscillator potential

As above, using the independent variable $z := (p_\theta + \mu p)/\beta$ for the oscillator potential $V = \beta \tan^2\theta/2$, one can reduce system (4.3) to the system

$$p_1'(z) = A(z)p_1 + B(z)\sqrt{f(z)}\,p_2, \qquad p_2'(z) = C(z)\sqrt{f(z)}\,p_1 - A(z)p_2 \tag{4.14}$$

with coefficients

$$A(z) = \frac{p}{f(f+1)}, \qquad B(z) = \frac{p}{\mu f^2} - \frac{\beta z + (2-\mu)p}{f(f+1)}, \qquad C(z) = \frac{\beta z - \mu p}{f(f+1)},$$

where $f(z) = \tan^2\theta = -\frac{\beta}{\mu}z^2 + 2\varepsilon$.

In the general case, the coefficients of system (4.14) are not rational due to the appearance of $\sqrt{f(z)}$. The same difficulty for the restricted two-body problem was overcome in [21] by the assumption $\varepsilon = 0$, when $\sqrt{f(z)} = \sqrt{-\beta/\mu}\,z$. On the other hand, it was noted in [22] that one can pass on to a second-order differential equation with rational coefficients.

Using the latter approach, one gets from (4.14) the following equation for $p_2(z)$:

$$p_2''(z) = \left(\frac{C'}{C} + \frac{f'}{2f}\right)p_2' + \left(\left(\frac{C'}{C} + \frac{f'}{2f}\right)A + A^2 + CBf - A'\right)p_2,$$

which can be reduced to equation (A.2) by the substitution $p_2(z) = y(z)\sqrt{C(z)}\,(f(z))^{1/4}$. Here

$$r(z) = \left(\frac{C'}{C} + \frac{f'}{2f}\right)A + A^2 + CBf - A' - \frac{1}{2}\left(\frac{C'}{C} + \frac{f'}{2f}\right)' + \frac{1}{4}\left(\frac{C'}{C} + \frac{f'}{2f}\right)^2.$$

Denote by $z_0 := p\mu/\beta$, $z_{1,2} := \pm\varkappa := \pm\sqrt{\mu(2\varepsilon+1)/\beta}$ and $z_{3,4} := \pm\lambda := \pm\sqrt{2\mu\varepsilon/\beta}$ zeros of functions $C(z)$, $f(z)+1$ and $f(z)$ respectively. Using computer calculations one gets

$$r(z) = \sum_{j=0}^{4}\left(\frac{\alpha_j}{(z-z_j)^2} + \frac{\beta_j}{z-z_j}\right) = O\left(\frac{1}{z^4}\right) \qquad \text{as} \quad z \to \infty, \tag{4.15}$$

where

$$\alpha_0 = \frac{3}{4}, \qquad \alpha_3 = \alpha_4 = -\frac{3}{16}, \qquad \beta_0 = \frac{3(\lambda^2 - \varkappa^2)p}{2((\lambda^2-\varkappa^2)^2p^2 - \lambda^2)},$$

$$\alpha_1 = \frac{\mu-1}{4\varkappa^2}(p(\mu-1)(\varkappa^2-\lambda^2) - \varkappa(\mu+1))(p(\varkappa^2-\lambda^2) - \varkappa),$$

$$\alpha_2 = \frac{\mu-1}{4\varkappa^2}(p(\mu-1)(\varkappa^2-\lambda^2) + \varkappa(\mu+1))(p(\varkappa^2-\lambda^2) + \varkappa),$$

$$\beta_1 = \frac{\mu-1}{4(\varkappa^2-\lambda^2)\varkappa^3}((\mu-1)(\lambda^2-3\varkappa^2)(\varkappa^2-\lambda^2)^2p^2 + 4\mu(\varkappa^2-\lambda^2)\varkappa^3 p - (\mu+1)\varkappa^2(\varkappa^2+\lambda^2)),$$

$$\beta_2 = \frac{\mu-1}{4(\varkappa^2-\lambda^2)\varkappa^3}((\mu-1)(3\varkappa^2-\lambda^2)(\varkappa^2-\lambda^2)^2p^2 + 4\mu(\varkappa^2-\lambda^2)\varkappa^3 p + (\mu+1)\varkappa^2(\varkappa^2+\lambda^2)),$$

$$\beta_3 = [8(\mu-1)^2(\varkappa^2-\lambda^2)^3p^3 - 8(3\mu-1)(\mu-1)\lambda(\varkappa^2-\lambda^2)^2p^2 + \lambda((5-8\mu^2)\lambda^2+3\varkappa^2) + (\varkappa^2-\lambda^2)(9\varkappa^2+(24\mu^2-16\mu-17)\lambda^2)p]/ [16\lambda(\varkappa^2-\lambda^2)((\varkappa^2-\lambda^2)p-\lambda)],$$

$$\beta_4 = [8(\mu-1)^2(\lambda^2-\varkappa^2)^3p^3 - 8(3\mu-1)(\mu-1)\lambda(\varkappa^2-\lambda^2)^2p^2 + \lambda((5-8\mu^2)\lambda^2+3\varkappa^2) + (\lambda^2-\varkappa^2)(9\varkappa^2+(24\mu^2-16\mu-17)\lambda^2)p]/ [16\lambda(\varkappa^2-\lambda^2)((\varkappa^2-\lambda^2)p+\lambda)]. \tag{4.16}$$

For $\mu = 1$ the function $r(z)$ has the form

$$r(z) = \frac{3}{4}\frac{\left(z_0^2-2\lambda^2\right)z^2+2\lambda^2z_0z+\lambda^2\left(\lambda^2-2z_0^2\right)}{(z-z_0)^2(z^2-\lambda^2)^2}. \tag{4.17}$$

In the hyperbolic case system (4.5) again is reduced to system (4.14), where now

$$A(z) = \frac{p}{f(1-f)}, \qquad B(z) = \frac{p}{\mu f^2} + \frac{\beta z+(2-\mu)p}{f(1-f)}, \qquad C(z) = \frac{\beta z-\mu p}{f(1-f)},$$

and $f(z) = \tanh^2\theta = -\frac{\beta}{\mu}z^2 + 2\varepsilon$.

By reasoning as above for the spherical case one can get equation (A.2) for the function $y(z) := p_2(z)\,(C(z))^{-1/2}\,(f(z))^{-1/4}$. Now the singular points are $z_0 := p\mu/\beta, z_{1,2} = \pm\varkappa, z_{3,4} = \pm\lambda, z_5 = \infty$ for

$$\varkappa := \sqrt{\frac{\mu}{\beta}(2\varepsilon-1)}, \qquad \lambda := \sqrt{\frac{2\mu\varepsilon}{\beta}}$$

and $r(z)$ is given by (4.15) with

$$\alpha_0 = \frac{3}{4}, \qquad \alpha_3 = \alpha_4 = -\frac{3}{16}, \qquad \beta_0 = \frac{3(\varkappa^2-\lambda^2)p}{2((\lambda^2-\varkappa^2)^2p^2-\lambda^2)},$$

$$\alpha_1 = \frac{\mu-1}{4\varkappa^2}(p(\mu-1)(\varkappa^2-\lambda^2)+\varkappa(\mu+1))(p(\varkappa^2-\lambda^2)+\varkappa),$$

$$\alpha_2 = \frac{\mu-1}{4\varkappa^2}(p(\mu-1)(\varkappa^2-\lambda^2)-\varkappa(\mu+1))(p(\varkappa^2-\lambda^2)-\varkappa),$$

$$\beta_1 = \frac{\mu-1}{4(\varkappa^2-\lambda^2)\varkappa^3}((\mu-1)(\lambda^2-3\varkappa^2)(\varkappa^2-\lambda^2)^2p^2 - 4\mu(\varkappa^2-\lambda^2)\varkappa^3p - (\mu+1)\varkappa^2(\varkappa^2+\lambda^2)),$$

$$\beta_2 = \frac{\mu-1}{4(\varkappa^2-\lambda^2)\varkappa^3}((\mu-1)(3\varkappa^2-\lambda^2)(\varkappa^2-\lambda^2)^2p^2 - 4\mu(\varkappa^2-\lambda^2)\varkappa^3p + (\mu+1)\varkappa^2(\varkappa^2+\lambda^2)), \tag{4.18}$$

$$\beta_3 = [8(\mu-1)^2(\varkappa^2-\lambda^2)^3p^3 + 8(3\mu-1)(\mu-1)\lambda(\varkappa^2-\lambda^2)^2p^2 + \lambda((8\mu^2-5)\lambda^2-3\varkappa^2) + (\varkappa^2-\lambda^2)(9\varkappa^2+(24\mu^2-16\mu-17)\lambda^2)p]/ [16\lambda(\varkappa^2-\lambda^2)((\varkappa^2-\lambda^2)p+\lambda)],$$

$$\beta_4 = [8(\mu-1)^2(\lambda^2-\varkappa^2)^3p^3 + 8(3\mu-1)(\mu-1)\lambda(\varkappa^2-\lambda^2)^2p^2 + \lambda((8\mu^2-5)\lambda^2-3\varkappa^2) + (\lambda^2-\varkappa^2)(9\varkappa^2+(24\mu^2-16\mu-17)\lambda^2)p]/ [16\lambda(\varkappa^2-\lambda^2)((\varkappa^2-\lambda^2)p-\lambda)].$$

For $\mu = 1$ the function $r(z)$ coincides with (4.17).

Lemma 4.3. *Suppose that the* $\alpha, \varepsilon, \mu, p \in \mathbb{R}$, $p \neq 0$, $\mu \neq 0, 1$, *and* $\varepsilon < -1/2$ *in the spherical case or* $\varepsilon < 1/2$ *in the hyperbolic case, then* $\alpha_i \notin \mathbb{R}$ *for* $i = 1, 2$.

Proof. Clearly, in both cases $\varkappa \in \mathrm{i}\mathbb{R}\backslash(0)$ and one gets

$$\mathrm{i}\,\mathrm{Im}\,\alpha_1 = -\mathrm{i}\,\mathrm{Im}\,\alpha_2 = \frac{\mu^2(1-\mu)p}{2\beta\varkappa} \neq 0. \qquad \square$$

5. Proof of nonintegrability

Lemma 5.1.

(i) Suppose that the assumptions of lemma 4.1 are valid. Then the identity component $\mathcal{G}_0$ *of the Galois group for equation (A.2) with* $r(z)$ *given by (4.9) and (4.10) is not Abelian.*

(ii) Suppose that the assumptions of lemma 4.2 are valid. Then the identity component $\mathcal{G}_0$ *of the Galois group for equation (A.2) with* $r(z)$ *given by (4.9) and (4.13) is not Abelian.*

Proof. We shall prove both claims of this lemma simultaneously. Here, there are six regular singular points of order 2: z_j, $j = 0, \ldots, 4$ and $z_5 = \infty$. The difference of exponents at points z_j, $j = 0, \ldots, 5$ is $\Delta_0 = \Delta_\infty = 2$, $\Delta_j = \sqrt{1+4\alpha_j}$, $j = 1, 2, 3, 4$ and due to lemma 4.1 it holds $\Delta_j \notin \mathbb{R}$, $j = 1, 2, 3, 4$. Therefore, the third case from lemma A.1 is impossible.

Consider the first case of lemma A.1. Here, one or two linear independent solutions $y_k(z)$ of (A.2) should exist such that $y_k'/y_k \in \mathbb{C}(z)$.

The rational function y_k'/y_k has no poles of order more than 1 since at such a pole the growth of y_k is exponential that is impossible in the Fuchsian case. Due to the same reason it should be $(y_k'/y_k)(z) \to 0$ as $z \to \infty$. This yields

$$\frac{y_k'(z)}{y_k(z)} = \sum_l \frac{\delta_l}{z - \tilde{z}_l}, \qquad \delta_l \in \mathbb{C}$$

and one can conclude that

$$y_k(z) = P_k(z) \prod_{j=0}^{4} (z - z_j)^{\rho_k^{(j)}}, \qquad P_k(z) \in \mathbb{C}[z], \tag{5.1}$$

where

$$\rho_k^{(0)} \in \left(-\tfrac{1}{2}, \tfrac{3}{2}\right), \qquad \rho_k^{(j)} \in \left(\tfrac{1}{2}(1+\Delta_j), \tfrac{1}{2}(1-\Delta_j)\right), \quad j = 1, \ldots, 4, \quad k = 1, 2.$$

Suppose first that there are two such linear independent solutions y_1 and y_2. Then due to lemma A.2 it holds $v(z) := y_1(z)y_2(z) \in \mathbb{C}(z)$.

Hence, possible exponents for $v(z)$ are $-1, 1, 3$ at the point z_0 and $1, 1 \pm \sqrt{1+4\alpha_j} \notin \mathbb{R}$ at points z_j, $j = 1, \ldots, 4$. The inclusion $v(z) \in \mathbb{C}(z)$ implies therefore only two possibilities:

$$v(z) = \frac{P(z)}{z - z_0} \prod_{j=1}^{4} (z - z_j) \quad \text{or} \quad v(z) = P(z) \prod_{j=1}^{4} (z - z_j), \qquad P(z) \in \mathbb{C}[z].$$

From (4.9) one can find that the exponents for (A.2) at ∞ are $-3/2$ and $1/2$. Consequently, the function $v(z)$ grows as $z \to \infty$ no faster than z^3. Thus, the only possibility (up to a constant nonzero multiple) for v is

$$v(z) = \frac{1}{z - z_0} \prod_{j=1}^{4} (z - z_j). \tag{5.2}$$

But direct computer calculations show that for (5.2) it holds

$$v''' - 4rv - 2r'v = \frac{p\widetilde{P}(z)}{(z-z_0)^3(z^2-\varkappa^2)(z^2-\lambda^2)},$$

where $\widetilde{P}(z)$ is a polynomial with the leading term $12\mathrm{i}(\varkappa^2-\lambda^2)z^6 = -48\mu z^6/\alpha \neq 0$ for the spherical case and $12(\lambda^2-\varkappa^2)z^6 = 48\mu z^6/\alpha \neq 0$ for the hyperbolic case. Hence, equation (A.3) cannot hold.

Thus, in case I of lemma A.1 there can be only one linear independent solution $y_1(z)$ of (A.2) such that $y_1'/y_1 \in \mathbb{C}(z)$. Since exponents at the points z_j, $j = 1, 2, 3, 4$ are not real one can conclude from lemma A.2 that in this case the group $\mathcal{G}$ is conjugate to the full triangular group, coincides with $\mathcal{G}_0$ and is not Abelian.

Now, using the Kovacic algorithm, we shall show that the second case of lemma A.1 cannot occur. Clearly, it holds $\operatorname{ord} z_j = 2$, $j = 0, \ldots, 5$ and $E_0 = E_\infty = (-2, 2, 6)$, $E_j = (2)$, $j = 1, \ldots, 4$.[1] Therefore one gets $d(e) = \frac{1}{2}(e_\infty - e_0) - 4$ and the maximal value for $d(e)$ is 0, which corresponds only to $e_0 = -2$, $e_\infty = 6$.

Thus, one should define

$$\Theta(z) := -\frac{1}{z-z_0} + \sum_{j=1}^{4} \frac{1}{z-z_j}$$

and verify the equality

$$\Xi(z) := \Theta'' + 3\Theta\Theta' + \Theta^3 - 4r\Theta - 2r' = 0. \qquad (5.3)$$

But computer calculations show that

$$\Xi(z) = \frac{pP_*(z)}{\prod_{j=0}^{4}(z-z_j)^2},$$

where $P_*(z)$ is a polynomial with the leading term $3\mathrm{i}(\varkappa^2-\lambda^2)z^6$ for the spherical case and $3(\lambda^2-\varkappa^2)z^6$ for the hyperbolic case. Thus, the second case of lemma A.1 cannot occur.

One can conclude from lemma A.1 that the differential Galois group for equation (A.2) with $r(z)$ given by (4.9), (4.10) or (4.9), (4.13) is either full triangular group (A.4) or $\mathrm{SL}_2(\mathbb{C})$. In both cases it coincides with its identity component $\mathcal{G}_0$ and is not Abelian. □

Note that due to (4.11) for $\mu = 1$ equation (A.2) with $r(z)$ given by (4.9), (4.10) or (4.9), (4.13) has linear independent solutions $y_1 = (z-z_0)^{3/2}$ and $y_2 = (z-z_0)^{-1/2}$. Therefore, for $\mu = 1$ the second case of lemma A.2 occurs and $\mathcal{G} = \mathbb{Z}_2$. Here, $\mathcal{G}_0$ is trivial that corresponds to the existence of the additional integral $p_1 \sin\theta + p_2 \cos\theta$ for the Hamiltonian function (4.1) and $p_1 \sinh\theta + p_2 \cosh\theta$ for the Hamiltonian function (4.2).

Lemma 5.2. *Suppose that assumptions of lemma 4.3 are valid. Then, the identity component $\mathcal{G}_0$ of the Galois group for equation (A.2) with $r(z)$ given by (4.15), (4.16) or (4.15), (4.18) is not Abelian.*

Proof. There are five regular singular points z_j, $j = 0, \ldots, 4$, of order 2 and the regular singular point $z_5 = \infty$ of order 0. One has $\Delta_0 = 2$, $\Delta_j = \sqrt{1+4\alpha_j}$, $j = 1, 2$, $\Delta_{3,4} = 1/2$ and due to lemma 4.3 $\Delta_j \notin \mathbb{R}$, $j = 1, 2$. Therefore the third case from lemma A.1 is impossible.

[1] For brevity we use the notation $E_j := E_{z_j}$.

Consider the first case of lemma A.1. Suppose that there are two linear independent solutions $y_k(z)$, $k = 1, 2$, of (A.2) such that $y_k'/y_k \in \mathbb{C}(z)$. By reasoning as in the proof of lemma 5.1 one can write them in form (5.1), where

$$\rho_k^{(0)} \in \left(-\tfrac{1}{2}, \tfrac{3}{2}\right), \rho_k^{(j)} \in \left(\tfrac{1}{2}(1+\Delta_j), \tfrac{1}{2}(1-\Delta_j)\right), \qquad j = 1, 2,$$
$$\rho_k^{(j)} \in \left(\tfrac{1}{4}, \tfrac{3}{4}\right), \quad j = 3, 4.$$

Besides it holds $v(z) := y_1(z)y_2(z) \in \mathbb{C}(z)$.

Hence, possible exponents for $v(z)$ are $-1, 1, 3$ at the point z_0; $1, 1 \pm \sqrt{1+4\alpha_j} \notin \mathbb{R}$ at points z_j, $j = 1, 2$; and $1/2, 1, 3/2$ at points z_j, $j = 3, 4$. The inclusion $v(z) \in \mathbb{C}(z)$ therefore implies only two possibilities:

$$v(z) = P(z)\prod_{j=1}^{4}(z - z_j) \quad \text{or} \quad v(z) = \frac{P(z)}{z - z_0}\prod_{j=1}^{4}(z - z_j), \quad P(z) \in \mathbb{C}[z].$$

Due to (4.15) the exponents for (A.2) at ∞ are 0 and -1; the function $v(z)$ grows as $z \to \infty$, no faster than z^2; therefore, no one of these possibilities can realize.

Thus in the case I of lemma A.1 there can be only one linear independent solution $y_1(z)$ of (A.2) such that $y_1'/y_1 \in \mathbb{C}(z)$. By reasoning as in the proof of lemma 5.1 one gets that in this case the group $\mathcal{G}_0$ is conjugate to the full triangular group and is not Abelian.

Check the possibility of the second case of lemma A.1 using the Kovacic algorithm. Clearly, it holds $\operatorname{ord} z_j = 2$, $j = 0, \ldots, 4$, $\operatorname{ord} \infty = 0$ and $E_0 = (-2, 2, 6)$, $E_{1,2} = (2)$, $E_{3,4} = (1, 2, 3)$, $E_\infty = (0, 2, 4)$. Therefore, a unique element $e \in E$ for which $d(e) \geqslant 0$ is $(-2, 2, 2, 1, 1, 4)$ and $d(-2, 2, 2, 1, 1, 4) = 0$.

Thus one should define

$$\Theta(z) := -\frac{1}{z - z_0} + \frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{2(z - z_3)} + \frac{1}{2(z - z_4)}$$

and verify equality (5.3).

But computer calculations show that

$$\Xi(z) = \frac{2p(2\mu+1)(\varkappa^2-\lambda^2)^2z^2 - 8p^2(\mu-1)(\varkappa^2-\lambda^2)^3z + 2p^2(\mu-1)(\varkappa^2-\lambda^2)^2 - 3\varkappa^2}{(z-z_3)(z-z_4)\prod_{j=0}^{2}(z-z_j)^2}.$$

for the spherical case and

$$\Xi(z) = \frac{-2p(2\mu+1)(\varkappa^2-\lambda^2)^2z^2 - 8p^2(\mu-1)(\varkappa^2-\lambda^2)^3z - 2p^2(\mu-1)(\varkappa^2-\lambda^2)^2 + 3\varkappa^2}{(z-z_3)(z-z_4)\prod_{j=0}^{2}(z-z_j)^2}.$$

for the hyperbolic case. Thus the second case of lemma A.1 cannot occur.

Now lemma A.1 implies that the differential Galois group for equation (A.2) with $r(z)$ given by (4.15), (4.16) or (4.15), (4.18) is either a full triangular group (A.4) or $\mathrm{SL}_2(\mathbb{C})$. In both cases it coincides with its identity component $\mathcal{G}_0$ and is not Abelian. □

Note that due to (4.17) for $\mu = 1$ equation (A.2) with $r(z)$ given by (4.15), (4.16) or (4.15), (4.18) has linear independent solutions

$$y_1(z) = \frac{(z^2-\lambda^2)^{3/4}}{(z-z_0)^{1/2}} \qquad \text{and} \qquad y_2(z) = \frac{(z^2-\lambda^2)^{1/4}(z_0z-\lambda^2)}{(z-z_0)^{1/2}}.$$

Therefore for $\mu = 1$ the second case of lemma A.2 occurs, $\mathcal{G} = \mathbb{Z}_4$ and $\mathcal{G}_0$ is trivial. This corresponds to the existence of the additional integrals described above.

Proof of theorems 2.1 and 2.2. Due to theorem 3.1 and the analysis in section 4 it is enough to show that the identity components of Galois groups for systems (4.7) and (4.14) described in section 4 are not Abelian.

Consider transformations of system (4.7) made in section 4.1. The base field for (4.7) is $\mathbb{C}(z)$. First, we reduced (4.7) to linear differential equation (4.8) of the second order that corresponds to the variable change

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \to \begin{pmatrix} p_2' \\ p_2 \end{pmatrix} = \begin{pmatrix} C(z) & -A(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

which is reversible over $\mathbb{C}(z)$. Therefore Galois groups for system (4.7) and equation (4.8) coincides. Then we came to equation (A.2) for the function $y(z) = p_2(z)/\sqrt{C(z)}$ with $r(z)$ given by (4.9). Since the function $\sqrt{C(z)}$ is algebraic the identity components of Galois groups for equations (4.8) and (A.2) are the same. One completes the proof for system (4.7) using lemmas 4.1 and 4.2.

The proof of theorem 2.2 is similar due to lemma 4.3 and since all transformations made in section 4.2 from system (4.14) to equation (A.2) are linear and their coefficients are algebraic functions. □

6. Conclusion

In the present paper we proved the complex nonintegrability of the reduced two-body problem in the spaces $\mathbf{S}^2$ and $\mathbf{H}^2$ for the Newton and the oscillator potentials. The main prerequisite for this proof was the possibility of reducing the system of normal variational equations to a linear differential equation of the second order with rational coefficients using a proper change of an independent variable. It is obvious that for a general central potential it could not be done.

Therefore, the problem of finding a nontrivial central potential corresponding to integrability of the two-body problem in constant curvature spaces or proving the absence of such potential in some more or less general class is open.

Appendix

We use the standard notations $\mathbb{C}(z)$ and $\mathbb{C}[z]$ for the field of rational functions and for the ring of polynomials, both with complex coefficients. Consider a linear second-order differential equation on the Riemannian sphere $\mathbf{P}^1(\mathbb{C})$:

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0, \qquad p(z), q(z) \in \mathbb{C}(z). \tag{A.1}$$

Any pole $z_0 \in \mathbb{C}$ of $p(z)$ or $q(z)$ is a singular point of equation (A.1). This point is a *regular singular point* for (A.1) if functions $(z-z_0)p(z)$ and $(z-z_0)^2 q(z)$ are holomorphic at z_0. One can find *exponents* $\rho^{(z_0)}$ of (A.1) at the point z_0 by the substitution $w(z) = (z-z_0)^{\rho^{(z_0)}}$ into (A.1) and keeping only leading terms as $z \to z_0$. This procedure gives a quadratic equation for $\rho^{(z_0)}$.

The same is also valid for the point $\infty \in \mathbf{P}^1(\mathbb{C})$ w.r.t. a variable $\zeta = 1/z$.

Equation (A.1) is *Fuchsian* iff all its singular points are regular.

Definition A.1. *A solution $\widetilde{w}(z)$ of equation (A.1) is called Liouvillian if there is a tower of differential fields*

$$\mathbb{C}(z) = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_m$$

with $\widetilde{w}(z) \in \mathcal{K}_m$ and for each $i = 1, \ldots, m$ it holds $\mathcal{K}_i = \mathcal{K}_{i-1}(v_i)$, where one of the following three possibilities holds:

(1) the function v_i is algebraic over $\mathcal{K}_{i-1}$,
(2) it satisfies $v_i' \in \mathcal{K}_{i-1}$,

(3) it satisfies $v_i'/v_i \in \mathcal{K}_{i-1}$.

The substitution

$$w(z) = \exp\left(-\frac{1}{2}\int p(z)\,\mathrm{d}z\right) y(z)$$

transforms equation (A.1) into the equation

$$y''(z) = r(z)y(z), \tag{A.2}$$

where $r(z) := -q(z) + \frac{1}{2}p'(z) + \frac{1}{4}p^2(z)$. If $y_1(z), y_2(z)$ are solutions of (A.1), then a direct calculation shows that

$$v'''(z) = 4rv'(z) + 2r'v(z) \tag{A.3}$$

for $v(z) := y_1(z)y_2(z)$. Equation (A.3) is called the second symmetric power of (A.2).

The differential Galois group $\mathcal{G}$ for (A.2) is an algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$ [28]. The following lemma [30] contains a classification of the possible groups $\mathcal{G}$.

Lemma A.1. *One and only one of the following four cases can occur.*

Case I. The group $\mathcal{G}$ is conjugate to a subgroup of the full triangular group

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{C}^*, b \in \mathbb{C} \right\}; \tag{A.4}$$

in this case equation (A.2) has a solution $y_1(z) \not\equiv 0$ such that $y_1'/y_1 \in \mathbb{C}(z)$.

Case II. The group $\mathcal{G}$ is conjugate to a subgroup of

$$\left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| \, c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| \, c \in \mathbb{C}^* \right\},$$

and case I does not hold. In this case equation (A.2) has a solution of the form $y_1(z) = \exp\left(\int \omega(z)\,\mathrm{d}z\right)$, where $\omega(z)$ is an algebraic function over $\mathbb{C}(z)$ of degree 2.

Case III. The group $\mathcal{G}$ is finite and cases I and II do not hold. In this case all solutions of (A.2) are algebraic and exponents of (A.2) are rational numbers at all points.

Case IV. $\mathcal{G} \simeq \mathrm{SL}_2(\mathbb{C})$ and equation (A.2) has no Liouvillian solutions.

A more precise information on the case I from the preceding lemma is contained in the following lemma.

Lemma A.2 (see proposition 4.2 from [31])**.** *Suppose that the case I from lemma A.1 occurs and therefore $\mathcal{G}$ is conjugate to a subgroup of the group T.*

(1) If equation (A.1) has a unique (up to a constant factor) solution $y_1(z) \not\equiv 0$ such that $y_1'/y_1 \in \mathbb{C}(z)$, then $\mathcal{G}$ is conjugate to a proper subgroup of the group T iff $y_1^m \in \mathbb{C}(z)$ for some $m \in \mathbb{N}$. In this case $\mathcal{G}$ is conjugate to

$$T_m = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in \mathbb{C}, a^m = 1 \right\},$$

where m is the smallest positive integer such that $y_1^m \in \mathbb{C}(z)$.

(2) *If equation (A.1) has two linear independent solutions* y_1, y_2 *such that* $y_j'/y_j \in \mathbb{C}(z)$, $j = 1, 2$, *then* $\mathcal{G}$ *is conjugate to a subgroup of the group*

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^* \right\}.$$

In this case $y_1 y_2 \in \mathbb{C}(z)$. *Finally,* $\mathcal{G}$ *is conjugate to a proper subgroup of the group* D *iff* $y_1^m \in \mathbb{C}(z)$ *for some* $m \in \mathbb{N}$. *In this case* $\mathcal{G}$ *is conjugate to a cyclic group of order* m, *where* m *is the smallest positive integer such that* $y_1^m \in \mathbb{C}(z)$.

We also need the Kovacic algorithm for case II from lemma A.1. It allows one to find a solution of equation (A.2) of the form $\exp\left(\int \omega(z)\,\mathrm{d}z\right)$, where $\omega(z)$ is an algebraic function of degree 2, or conclude that such a solution does not exist.

Let $r(z) = s(z)/t(z)$, where $s(z)$ and $t(z)$ are relatively prime polynomials and $t(z)$ is monic, i.e. the coefficient of its leading term equals 1. We denote $\Sigma' := (c \in \mathbb{C} |\, t(c) = 0)$ and $\Sigma := \Sigma' \cup (\infty)$.

Let an order $\operatorname{ord} c$ of $c \in \Sigma'$ be the multiplicity of c as a root of $t(z)$ and $\operatorname{ord}\infty := \max(0, 4 + \deg s - \deg t)$. If $c \in \Sigma$ such that $\operatorname{ord} c = 1$ or $\operatorname{ord} c = 2$, one can find an expansion

$$r(z) = \frac{\alpha_c}{(z-c)^2} + O\left(\frac{1}{z-c}\right) \quad \text{as} \quad z \to c \quad \text{for} \quad c \in \Sigma'$$

and

$$r(z) = \frac{\alpha_\infty}{z^2} + O\left(\frac{1}{z^3}\right) \quad \text{as} \quad z \to \infty \quad \text{for} \quad c = \infty.$$

In these cases $\Delta_c := \sqrt{1 + 4\alpha_c}$ is the difference of exponents for (A.2) at $z = c$. Evidently $\alpha_c = 0$ and $\Delta_c = 1$ if $\operatorname{ord} c = 1$.

Step 1. For every $c \in \Sigma$, we define a finite set E_c in the following way.
If $\operatorname{ord}\infty = 0$, then put $E_\infty = (0, 2, 4)$.
If $\operatorname{ord} c = 1$, then put $E_c = (4)$ for $c \neq \infty$ and $E_\infty = (0, 2, 4)$.
If $\operatorname{ord} c = 2$, then put

$$E_c = (2, 2(1 + \Delta_c), 2(1 - \Delta_c)) \cap \mathbb{Z}.$$

If $\operatorname{ord} c = k > 2$, then put $E_c = (k)$ for $c \neq \infty$ and $E_\infty = (4 - k)$.

Step 2. For each element e of the set

$$E := \prod_{c \in \Sigma} E_c,$$

we compute the number

$$d(e) := \frac{1}{2}\left(e_\infty - \sum_{c \in \Sigma'} e_c\right),$$

where e_c is a component of an array e from E_c. We select those elements $e \in E$ for which $d(e) \in \mathbb{Z}_+ := \mathbb{N} \cup (0)$. If there are no such elements, then the case II from lemma A.1 cannot occur.

Step 3. For each element $e \in E$ selected on the previous step we define

$$\Theta(z) := \frac{1}{2} \sum_{c \in \Sigma'} \frac{e_c}{z - c},$$

and search for a monic polynomial $P(z)$ of degree $d(e)$ satisfying the following equation:

$$P''' + 3\Theta P'' + (3\Theta^2 + 3\Theta' - 4r)P' + (\Theta'' + 3\Theta\Theta' + \Theta^3 - 4r\Theta - 2r')P = 0.$$

If such a polynomial exists, then equation (A.2) has a solution of the form $\exp\left(\int \omega(z)\,\mathrm{d}z\right)$, where

$$\omega^2 - \psi\omega + \frac{1}{2}\psi' + \frac{1}{2}\psi^2 - r = 0, \qquad \psi = \psi(z) := \Theta + \frac{P'}{P}.$$

If such a polynomial P does not exist, then case II from lemma A.1 does not occur.

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